Week 9 Summary: Graph Representations, BFS, DFS, and Topological Sort

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1. Graph Representations: Adjacency List vs. Adjacency Matrix

For a graph $G = (V, E)$ with $|V|$ vertices and $|E|$ edges:

Operation	Adjacency List	Adjacency Matrix	Preferred Regime
Memory	$\Theta(\|V\| + \|E\|)$	$\Theta(\|V\|^2)$	List for sparse graphs ($\|E\| \ll \|V\|^2$)
Edge query $(u, v) \in E$	$O(\deg(u))$	$O(1)$	Matrix for dense edge queries
Iterate neighbors of u	$\Theta(\deg(u))$	$\Theta(\|V\|)$	List for BFS and DFS

2. Breadth-First Search (BFS) and Unweighted Shortest Paths

Starting from source s, BFS maintains a FIFO queue and discovers vertices in non-decreasing hop distance $d(s, v)$. Every vertex enters the queue at most once and each adjacency list is scanned once, giving $\Theta(|V| + |E|)$ total time. Recording 'parent[v] = u' when v is first discovered builds a shortest-path tree whose root-to-v path has minimum edge count.

3. Depth-First Search (DFS), Edge Classification, and Cycle Detection

DFS colors vertices white (unvisited), gray (active on the recursion stack), and black (finished), recording discovery time $d[u]$ and finish time $f[u]$.

- **Parenthesis Theorem:** Intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are either disjoint or one is strictly nested inside the other.
- **Cycle Theorem:** A directed graph G has a directed cycle if and only if a DFS traversal encounters a back edge (u, v) pointing to a gray vertex v.

4. Topological Ordering of a DAG

A directed acyclic graph (DAG) admits a linear ordering of vertices where every edge (u, v) places u before v.

1. **Kahn's Algorithm:** Track in-degrees, enqueue all vertices with $\text{indeg}(u) = 0$, pop u, append to output, and decrement $\text{indeg}(v)$ for each outgoing edge (u, v). If fewer than $|V|$ vertices are emitted, a cycle exists. Runs in $\Theta(|V| + |E|)$.
2. **DFS Finish Order:** Reversing the order in which vertices turn black also produces a valid topological sort in $\Theta(|V| + |E|)$.

Worked Check: Zanzibar Prerequisite DAG

With edges $(\text{Python}, \text{Z5007})$, $(\text{DiscreteMath}, \text{Z5007})$, $(\text{Z5007}, \text{ML})$, $(\text{Z5007}, \text{BigData})$, $(\text{ML}, \text{Thesis})$, $(\text{BigData}, \text{Thesis})$, initial zero in-degree vertices are $\{\text{Python}, \text{DiscreteMath}\}$. Kahn's queue emits Python, DiscreteMath, Z5007, ML, BigData, Thesis.