Week 10 Summary: Shortest Paths (Dijkstra and Bellman-Ford)

Z5007 Programming and Data Structures · MTech DSAI · Odd Semester 2026-27 · Dr. Innocent Nyalala

1. Relaxation and Optimal Substructure

Every subpath of a shortest path is itself a shortest path. Shortest-path algorithms initialize $d[s] = 0$ and $d[v] = \infty$ for $v \neq s$, then repeatedly relax edges (u, v) of weight $w(u, v)$:

$$\text{if } d[u] + w(u,v) < d[v]: \quad d[v] \leftarrow d[u] + w(u,v), \quad \text{parent}[v] \leftarrow u.$$

2. Dijkstra's Algorithm ($w \geq 0$)

When all edge weights are non-negative, popping the unvisited vertex u with smallest $d[u]$ from a binary min-heap guarantees $d[u] = \delta(s, u)$ permanently.

- **Lazy Deletion in Python `heapq`:** Push $(d[v], v)$ whenever relaxation succeeds and discard stale popped pairs with $d > d[u]$.
- **Complexity:** Up to $|E|$ heap pushes and $|V|$ extracts yield $O((|V| + |E|) \log |V|)$ time and $O(|V| + |E|)$ space.
- **Failure with Negative Edges:** A later negative edge can lower an already-settled vertex's true distance, breaking the greedy invariant.

3. Bellman-Ford Algorithm (Arbitrary Weights & Negative Cycles)

Bellman-Ford relaxes all $|E|$ edges across $|V|-1$ passes in $O(|V||E|)$ time. Because any simple shortest path has at most $|V| - 1$ edges, all true shortest distances converge by pass $|V| - 1$ whenever no negative cycle is reachable from s. Running one extra pass checks whether any edge still relaxes: if $d[u] + w(u, v) < d[v]$, a negative-weight cycle is reachable.

Worked Zanzibar Routing Trace

From Stone Town, edges (StoneTown, Mbweni, 6), (StoneTown, Fumba, 18), (Mbweni, Bweleo, 11), and (Fumba, Bweleo, 5) settle Mbweni at 6 km first, then relax Bweleo to 17 km via StoneTown → Mbweni → Bweleo (beating StoneTown → Fumba → Bweleo at 23 km).

Algorithm	Edge Weights	Time Complexity	Negative Cycle Detection
BFS	Unweighted ($w = 1$)	$O(\|V\| + \|E\|)$	Not applicable
Dijkstra (binary heap)	$w(u, v) \geq 0$	$O((\|V\| + \|E\|) \log \|V\|)$	Invalid input
Bellman-Ford	Any real $w(u, v)$	$O(\|V\|\|E\|)$	Yes ($\|V\|$-th pass)